

The Minimal Automorphism-Free Tree

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Abstract

A finite tree T with $|V(T)| \geq 2$ is called *automorphism-free* if there is no non-trivial automorphism of T . Let \mathcal{AFT} be the poset with the element set of all finite automorphism-free trees (up to graph isomorphism) ordered by $T_1 \preceq T_2$ if T_1 can be obtained from T_2 by successively deleting one leaf at a time in such a way that each intermediate tree is also automorphism-free. In this paper, we prove that \mathcal{AFT} has a unique minimal element. This result gives an affirmative answer to the question asked by Rupinski in [1].

1 Introduction

In this paper, every graph is finite and simple. For a graph G , a bijection $\phi : V(G) \rightarrow V(G)$ is an *automorphism* of G if $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(G)$. For example, the identity function on $V(G)$ is an automorphism of G . We call this identity function the *trivial* automorphism.

For a tree T with $|V(T)| \geq 2$, we say T is *automorphism-free* if there is no non-trivial automorphism of T . For instance, the following graph E_7 in Figure 1 is automorphism-free. It is easy to check that there are no automorphism-free trees with fewer than 7 vertices.

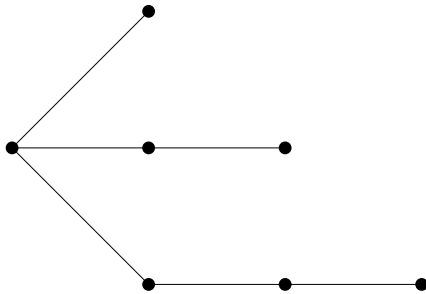


Figure 1: E_7

Let \mathcal{AFT} be the poset (partially ordered set) with the element set of all finite automorphism-free trees (up to graph isomorphism) ordered by $T_1 \preceq T_2$ if T_1 can be obtained from T_2 by successively deleting one leaf at a time in such a way that each intermediate tree is also automorphism-free.

In this paper, we prove that \mathcal{AFT} has a unique minimal element, namely E_7 .

1.1. *Let T be a minimal element of the poset \mathcal{AFT} . Then T is isomorphic to E_7 .*

Equivalently,

1.2. *Every automorphism-free tree T can be obtained from E_7 by successively adjoining a leaf at a time in such a way that each intermediate tree is also automorphism-free.*

or,

1.3. *For every automorphism-free tree T , E_7 can be obtained from T by successively deleting a leaf at a time in such a way that each intermediate tree is also automorphism-free.*

This result gives an affirmative answer to the question asked by Rupinski in [1]. First, we start with some definitions. A *component* of a graph G is a maximal non-null subgraph of G . For a vertex u of a graph G , $G \setminus u$ denotes the graph obtained from G by deleting the vertex u (deleting all the edges incident with u as well). For an edge uv of a graph G , $G \setminus uv$ denotes the graph obtained from G by deleting the edge uv (not deleting the vertex u or v). For a vertex set $S \subseteq V(G)$ of a graph G , $G|S$ denotes the subgraph of G induced by S . For a tree T , a *leaf* l is a vertex of degree one in T , and $p(l)$ denotes the (unique) neighbor of l in T . For a path P , the *length* of P is the number of edges in P . For a tree T and $u, v \in V(T)$, $\text{dist}_T(u, v)$ is the length of the (unique) path from u to v in T . For each $v \in V(T)$, $d_T(v) := \max_{u \in V(T) \setminus \{v\}} \text{dist}_T(u, v)$. We say $v \in V(T)$ is a

center of T if $d_T(v) \leq d_T(u)$ for every $u \in V(T)$, and the *radius* $r(T)$ of T denotes the number $d_T(v)$.

For the proof of 1.1, we look at a minimal element T of $\mathcal{AF}\mathcal{T}$. In T , we choose special leaves l_1 and l_2 by certain methods, and use the fact that both $T \setminus l_1$ and $T \setminus l_2$ have non-trivial automorphisms. From this, we find various properties that T must have. For instance, we prove that T must have two centers, and $T \setminus l_1$ must have exactly one center, and $T \setminus l_2$ must have two centers, etc. Eventually we prove that T must be isomorphic to E_7 .

2 Main proof

The following is an easy lemma about centers in a tree. We omit the proof.

2.1. *Let T be a tree with $|V(T)| \geq 2$. Let l be a leaf of T and let ϕ be an automorphism of T .*

- (1) *If u and v are distinct centers of T , then $uv \in E(T)$. In particular, there are at most two centers of T .*
- (2) *If u and v are distinct centers of T , then every path of length $r(T)$ from u contains v , and vice versa.*
- (3) *If u is the unique center of T , then $\phi(u) = u$.*
- (4) *If u and v are distinct centers of T , then ϕ either fixes u and v or switches them.*
- (5) *If u is the unique center of T , then it is a center of $T \setminus l$ as well.*
- (6) *If u and v are distinct centers of T , then every center of $T \setminus l$ is either u or v .*

For a given tree T with $|V(T)| \geq 2$ and a vertex u of T , we say a leaf $l (\neq u)$ is a *special leaf* with respect to T and u if the following statement holds for l .

Let P be the path from u to l in T and number the vertices of P as $v_1 (= u), \dots, v_m (= l)$ in order. For each $i = 1, \dots, m-1$, let C_i be the component of $T \setminus v_i$ containing l . Then for every component C of $T \setminus v_i$ not containing u , $|V(C)| \geq |V(C_i)|$.

2.2. *Let T be a tree with $|V(T)| \geq 2$ and let $u \in V(T)$. Then, there exists a special leaf with respect to T and u .*

Proof. We proceed by induction on $|V(T)|$. It is easy to see that the statement holds for $|V(T)| = 2$. Consider all neighbors of u . Each one is in its own component of $T \setminus u$. Among those components, we take one with the least number of vertices. (If there is more than one smallest component, just pick any one of them.) Let C be the component of $T \setminus u$ we chose and let v be the neighbor of u in C . Now, look at all children of v (the neighbors of v in C). If there are no children of v , then v is a special leaf with respect to T and u we are looking for. Therefore we may assume $|V(C)| \geq 2$. Then from the induction hypothesis, there is a special leaf l with respect to C and v . Then, it is easy to check that l is a special leaf with respect to T and u as well. This proves 2.2. ■

2.3. Let T be a minimal tree in the poset \mathcal{AFT} . Then T has two centers.

Proof. For the sake of contradiction, suppose T has only one center u . Let l_1 be the special leaf with respect to T and u . Let $T' = T \setminus l_1$ and take a non-trivial automorphism ϕ of T' . By 2.1 (5), u is a center of T' as well.

(1) ϕ does not fix u . In particular, there is another center of T' .

Suppose ϕ fixes u . Notice that ϕ does not fix $p(l_1)$ because otherwise we can extend ϕ to a non-trivial automorphism of T by assigning $\phi(l_1) = l_1$. In particular, $u \neq p(l_1)$. Let P be the path from u to $p(l_1)$ in T' , and number the vertices of P as $v_1(=u), \dots, v_m(=p(l_1))$ in order ($m \geq 2$). Let k be the largest number such that $\phi(v_k) = v_k$ (and so, $k \leq m-1$ and $\phi(v_{k+1}) \neq v_{k+1}$).

In $T' \setminus v_k$, let C_1 be the component containing v_{k+1} (this component contains $p(l_1)$ as well), and let C_2 be the component containing $\phi(v_{k+1})$. Clearly, C_1 and C_2 are different since v_{k+1} and $\phi(v_{k+1})$ are both neighbors of v_k .

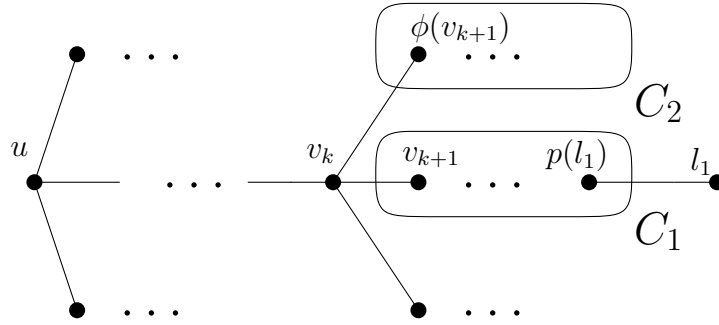


Figure 2: C_1 and C_2

Notice that $\phi|_{V(T' \setminus v_k)}$ is an automorphism of $T' \setminus v_k$ since ϕ fixes v_k . This implies C_1 and C_2 are isomorphic. In particular, $|V(C_1)| = |V(C_2)|$. But back in T , $T|(V(C_1) \cup \{l_1\})$ and C_2 are two components of $T \setminus v_k$, and $|V(C_1) \cup \{l_1\}| > |V(C_2)|$. This contradicts the definition of l_1 . This proves (1).

Let v be the center of T' different from u .

(2) $d_T(v) = r(T) + 1$, and there is a unique path of length $d_T(v)$ from v in T , namely the path from v to l_1 .

In T , u is the unique center. But in T' , both u and v are centers. Therefore

$$d_T(v) - 1 \geq r(T) = d_T(u) \geq d_{T'}(u) = d_{T'}(v) \geq d_T(v) - 1.$$

In particular, $d_T(v) - 1 = d_{T'}(v)$. This implies there is no path of length $d_T(v)$ from v in T' . Since there is a path of length $d_T(v)$ from v in T , namely the path from v to l_1 , it must be unique. This proves (2).

(3) l_1 is not a neighbor of u .

Suppose l_1 is a neighbor of u . Then from (2),

$$r(T') = d_{T'}(v) = d_T(v) - 1 = \text{dist}_T(v, l_1) - 1 = 1.$$

Since T' has two centers and $r(T') = 1$, $|V(T')| = 2$ and $|V(T)| = 3$. But this is impossible since there is no automorphism-free tree with three vertices. This proves (3).

(4) $p(l_1)$ has degree two in T .

If there exists another child w of $p(l_1)$ in T , then the path from v to w is another path of length $d_T(v)$ from v in T , which is impossible by (2). Therefore l_1 is the unique child of $p(l_1)$ in T . This proves (4).

Let T_u and T_v be the two components of $T \setminus uv$ containing u and v , respectively. Note that ϕ switches u and v by (1). And $T_u \setminus l_1$ is isomorphic to T_v by ϕ . Let l_2 be a special leaf with respect to T_v and v (l_2 exists since $V(T_v) \geq 2$). Let $T'' = T \setminus l_2$ and take a non-trivial automorphism ψ of T'' .

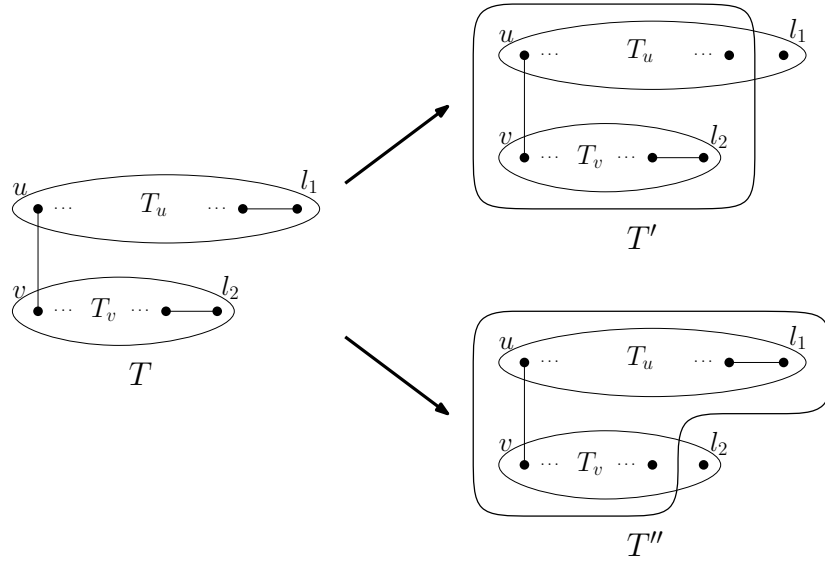


Figure 3: T' and T''

(5) u is a center of T'' , and v is not.

Again, u is a center of T'' by 2.1 (5). But, v is not a center of T'' because from (2),

$$d_{T''}(v) \geq \text{dist}_{T''}(v, l_1) = r(T) + 1 > d_T(u) \geq d_{T''}(u).$$

This proves (5).

(6) ψ does not fix v . Moreover, $\psi(V(T_v \setminus l_2)) \subseteq V(T_u \setminus u)$.

Suppose ψ fixes v . Then u is fixed as well because among the neighbors of v , u is the unique center of T'' (although u might not be the unique center of T''). Since both u and v are fixed by ψ , $\psi(V(T_v)) = V(T_v)$. Therefore $\psi|_{V(T_v \setminus l_2)}$ is a non-trivial automorphism of $T_v \setminus l_2$ (otherwise we can extend ψ to a non-trivial automorphism of T by assigning $\psi(l_2) = l_2$). Then by the same argument as in (1), this contradicts the definition of l_2 .

Since v is adjacent to a center of T'' , so is $\psi(v)$. And v is the unique such vertex in $V(T_v \setminus l_2)$. Therefore $\psi(v)$ does not belong to $V(T_v \setminus l_2)$ because $\psi(v) \neq v$. Also, every member of $\psi(V(T_v \setminus l_2))$ does not belong to $V(T_v \setminus l_2)$ either because $T_v \setminus l_2$ is a component of $T'' \setminus u$ containing v . Therefore $\psi(V(T_v \setminus l_2)) \subseteq V(T_u \setminus u)$. This proves (6).

Let C be the component of $T'' \setminus u$ containing $\psi(V(T_v \setminus l_2))$. Let $n = |V(T_v)|$.

(7) $|V(C)|$ is either n or $n - 1$.

Since $\psi(V(T_v \setminus l_2)) \subseteq V(C)$, $|V(C)| \geq n - 1$. Recall that $|V(T_u)| - 1 = |V(T_u \setminus l_1)| = |V(T_v)| = n$ and $V(C)$ is a subset of $V(T_u \setminus u)$. Therefore

$$|V(C)| \leq |V(T_u \setminus u)| = |V(T_u)| - 1 = n.$$

This proves (7).

(8) $|V(C)| = |V(T_u \setminus u)| = n$. In particular, the degree of u is two (both in T'' and T), and $T'' \setminus u$ consists of two components $T_u \setminus u (= C)$ and $T_v \setminus l_2$.

From (3), u has no neighbor which is a leaf (both in T and T''). In particular, every component of $T'' \setminus u$ has more than one vertex. Note that the union of all components of $T'' \setminus u$ different from $T_v \setminus l_2$ has size $|V(T_u \setminus u)| = n$. Since C is one of them whose size is at least $n - 1$, there cannot be another one (and so, $|V(C)| = n$). Therefore u has degree two, and this proves (8).

Notice that the degree of v in T is also two since $v = \phi(u)$.

(9) ψ does not fix u . In particular, T'' has two centers u and $\psi(u)$, and $T_v \setminus l_2 \cong T_u \setminus u \setminus \psi(u)$.

By (5), v is not a center of T'' and it is adjacent to a center of T'' . Therefore $\psi(v)$ has the same property in T'' . But $\psi(v)$ is not adjacent to the center u , because if it is, then $T_u \setminus u$ is isomorphic to $T_v \setminus l_2$ and hence, $|V(C)| = |V(T_v \setminus l_2)| = n - 1$. This is impossible by (8).

Therefore $\psi(v)$ is adjacent to another center, and this also implies $\psi(u) \neq u$. Since two centers are adjacent, $\psi(u)$ must be the neighbor of u different from v . Since $\psi(u)$ also has degree two, the neighbor of $\psi(u)$ different from u is $\psi(v)$. And $T_v \setminus l_2$ is isomorphic to $T_u \setminus u \setminus \psi(u)$. This proves (9).

Note that $T_u \setminus l_1 \cong T_v$ by ϕ , and $T_u \setminus u \setminus \psi(u) \cong T_v \setminus l_2$ by ψ .

(10) T_u and T_v are paths.

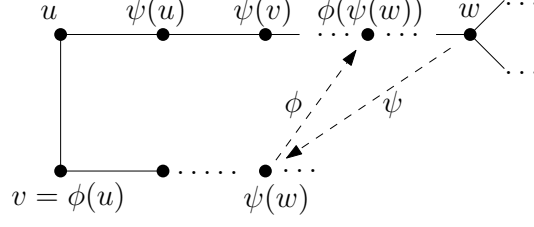


Figure 4: ϕ and ψ

It is enough to show that T_u is a path since $T_v \cong T_u \setminus l_1$. Suppose there exists a vertex of degree at least three in T_u . Choose such a vertex $w \in V(T_u)$ with $\text{dist}_T(u, w)$ as small as possible. Then, $\phi(\psi(w))$ has degree at least three in T_u as well. To see this, first observe that $\psi(w) \in V(T_v) \setminus \{v, l_2\}$ has degree at least three since $w \in V(T_u) \setminus \{u, \psi(u), \psi(v)\}$, and $T_u \setminus u \setminus \psi(u) \cong T_v \setminus l_2$. Therefore $\phi(\psi(w)) \in V(T_u)$ has degree at least three since $T_v \cong T_u \setminus l_1$. But then,

$$\text{dist}_T(u, \phi(\psi(w))) = \text{dist}_T(\phi^{-1}(u), \psi(w)) = \text{dist}_T(v, \psi(w)) = \text{dist}_T(\psi^{-1}(v), w) < \text{dist}_T(u, w).$$

This contradicts our choice of w . This proves (10).

Since both u and v have degree two in T , T is a path by (10). This contradicts the fact that T is automorphism-free. Therefore T has two centers. This proves 2.3. ■

Proof of 1.1. By 2.3, T has two centers u and v . Let T_u and T_v be the two components of $T \setminus uv$ containing u and v , respectively.

Let l_1 be a special leaf with respect to T_u and u and let l_2 be a special leaf with respect to T_v and v . Let x be the shortest distance from u to a vertex in T_u whose degree in T is at least three. If there is no such vertex, then set x as ∞ . Similarly, let y be the shortest distance from v to a vertex in T_v whose degree in T is at least three.

Without loss of generality, we may assume $|V(T_u)| \geq |V(T_v)|$. And further we may assume if $|V(T_u)| = |V(T_v)|$ then $x \geq y$ by switching u and v if necessary. We first consider $T' = T \setminus l_1$. Let ϕ be a non-trivial automorphism of T' .

(1) u and v are centers of T' .

By 2.1 (2), $d_{T'}(u) = d_T(u)$ since every path of length $d_T(u)$ from u in T does not contain l_1 . And by 2.1 (6), v is a center of T' since $d_{T'}(v) \leq d_T(v) = d_T(u) = d_{T'}(u)$. Again by 2.1 (6), if there is a center of T' different from v , then it must be u . For the sake of contradiction, suppose v is the unique center of T' . Then $p(l_1)$ is a leaf in T' by the same argument as in (4) in the proof of 2.3. Then, ϕ does not fix u since otherwise it contradicts the definition of l_1 , by the same argument as in (6) in the proof of 2.3. Therefore in $T' \setminus v$, the component $T_u \setminus l_1$ is isomorphic to another component C of $T' \setminus v$. Note that

$$|V(C)| = |V(T_u \setminus l_1)| = |V(T_u)| - 1.$$

Since $V(C)$ is a subset of $V(T_v \setminus v)$,

$$|V(T_v)| \geq |V(C)| + 1 = |V(T_u)|.$$

Together with our assumption $|V(T_u)| \geq |V(T_v)|$, $|V(T_v)| = |V(T_u)|$. Moreover $T' \setminus v$ has exactly two components, namely C and $T_u \setminus l_1$. In particular, v has degree two in T .

Next, there exists a vertex in $V(T_v)$ whose degree in T is at least three because otherwise T_v is a path, and hence so is T' , and so is T because $p(l_1)$ has degree two in T . But then, $y = x + 1$ by ϕ , and this contradicts our assumption that $x \geq y$ if $|V(T_u)| = |V(T_v)|$. Therefore v is not the unique center of T' . This implies u is a center of T' as well. This proves (1).

(2) ϕ switches u and v , and $|V(T_v)| = |V(T_u)| - 1$.

Again, if ϕ fixes u , then ϕ fixes v as well and this contradicts our choice of l_1 . Since ϕ switches u and v , T_v and $T_u \setminus l_1$ are isomorphic. In particular, $|V(T_v)| = |V(T_u)| - 1$. This proves (2).

Now we consider $T'' = T \setminus l_2$. Let ψ be a non-trivial automorphism of T'' .

(3) ψ does not fix v , and u is the unique center of T'' .

By the same argument as in the proof of (1), u is a center of T'' , and if there is another one, then it must be v . Suppose ψ fixes v . Then, again u is fixed as well and this contradicts the definition of l_2 by the same argument as in (6) in the proof of 2.3. Therefore ψ does not fix v .

Now, suppose v is a center of T'' . Since ψ does not fix v , it switches u and v and $T_v \setminus l_2$ is isomorphic to T_u . But then, $|V(T_v \setminus l_2)| = |V(T_v)| - 1 = |V(T_u)| - 2 \neq |V(T_u)|$, a contradiction. This proves (3).

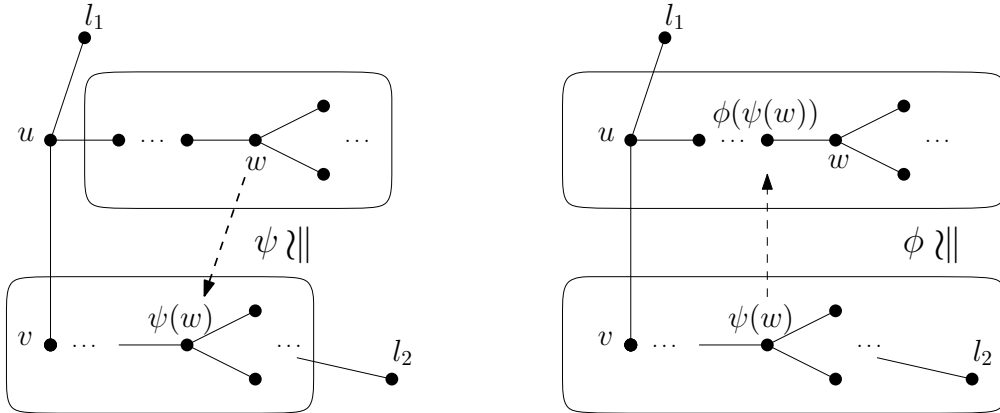


Figure 5: ϕ and ψ

Since u is the unique center of T'' and ψ does not fix v , the component $T_v \setminus l_2$ of $T'' \setminus u$ is isomorphic to another component C of $T'' \setminus u$. Note that the union of all components of $T'' \setminus u$ different from $T_v \setminus l_2$ is exactly $T_u \setminus u$. And C has size $|V(T_v)| - 1 = |V(T_u)| - 2$. This implies that there are exactly three components of $T'' \setminus u$, namely $T_v \setminus l_2$, C , and the third one with

a single vertex. Therefore u has a neighbor of degree one, and this implies that l_1 is a neighbor of u .

Now, $T_v \cong T_u \setminus l_1$ by ϕ , and $T_u \setminus u \setminus l_1 \cong T_v \setminus l_2$ by ψ .

(4) $T_u \setminus l_1$ and T_v are paths.

It is enough to show that $T_u \setminus l_1$ is a path since $T_v \cong T_u \setminus l_1$ by ϕ . Suppose there exists a vertex of degree at least three in $T_u \setminus l_1$. Choose such a vertex $w \in V(T_u \setminus l_1)$ with $\text{dist}_T(u, w)$ as small as possible. Then, $\phi(\psi(w))$ has degree at least three in $T_u \setminus l_1$ as well. To see this, first observe that $\psi(w) \in V(T_v \setminus l_2)$ has degree at least three in T_v since $w \in V(T_u \setminus u)$ and $T_u \setminus u \setminus l_1 \cong T_v \setminus l_2$. Therefore $\phi(\psi(w)) \in V(T_u \setminus l_1)$ has degree at least three in $T_u \setminus l_1$ since $T_u \setminus l_1 \cong T_v$. But then,

$$\begin{aligned} \text{dist}_T(u, \phi(\psi(w))) &= \text{dist}_T(\phi^{-1}(u), \psi(w)) = \text{dist}_T(v, \psi(w)) \\ &= \text{dist}_T(\psi^{-1}(v), w) = \text{dist}_T(\psi(v), w) < \text{dist}_T(u, w). \end{aligned}$$

This contradicts our choice of w . This proves (4).

By (4), T is a tree with a unique vertex of degree three, namely u , and one of the three components of $T \setminus u$ consists of a single vertex, namely l_1 , and the other two components T_v and $T_u \setminus u \setminus l_1$ are paths. Let $|V(T_u \setminus u \setminus l_1)| = k$; then $|V(T_v)| = k + 1$ by (2). Finally if $k > 2$, then deleting the leaf of T in $V(T \setminus u \setminus l_1)$ yields another automorphism-free tree, and if $k = 1$, then $|V(T)| = 5$, and so T is not automorphism-free. Therefore $k = 2$, and T is isomorphic to E_7 . This proves 1.1. ■

References

- [1] A. Rupinski, “Does this poset have a unique minimal element?”
 < <http://mathoverflow.net/questions/117151> >